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MATHEMATICS

Bilinear Systems

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INTRODUCTION

We propose a new class of dynamical systems consisting of many strongly interacting components. They admit an exact reduction (of the type of a self-consistent field) to simpler systems, whose dimension does not depend on the number of components. The interaction is not assumed to be small, which distinguishes this approach from all known ones and makes it possible to simulate complicated types of kinetics.

Consider a dynamical system consisting of a large number N of identical components:

$$\frac{dx_i}{dt} = a(x_i) + \sum_{k=1}^{N} b(x_i, x_k). \tag{1}$$

The mechanism of pairwise interactions in the case of bilinear systems is specified by a vector function $b(x, y)$ of two vector arguments, x and y. This function is the same for any pair of components x_i and x_k.

The number N characterizes the degree of atomicity (block nature) of the complex system under study. Typical values of N are huge, for example, the number of stars in a galaxy (10^{11}) or the number of gas molecules in a volume of 1 cm^3 (10^{16}). The range of N in biological systems is also wide (from hundreds to millions of components). A common feature of all these systems is that neither modern computers nor those to appear in the near future would be powerful enough to handle them.

The basic idea of the approach suggested can be illustrated by examples of bilinear systems. Moreover, this case is of interest in itself.

System (1) is called bilinear if $a(x)$ and $b(x, y)$ are linear functions of their arguments.

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At the first step, we only use the linearity of $b(x, y)$ in its second (external) argument. Introducing the new (aggregate) variable

$$X = \sum_{k=1}^{N} x_k, \tag{2}$$

we obtain an equation for x_i:

$$\frac{dx_i}{dt} = a(x_i) + b(x_i, X). \tag{3}$$

This equation has a great advantage, since it shows that x_i responds to the entire X rather than to each component x_k separately, as could be concluded at first glance from system (1).

The next step is of decisive importance. Since $a(x)$ and $b(x, y)$ are linear functions of x, we add all equations in (3) to obtain an equation for X alone, not involving other variables:

$$\frac{dx}{dt} = a(X) + B(X, X). \tag{4}$$

For bilinear systems, the aggregate variable X turns out to satisfy a reference equation.

Thus, we have obtained the system

$$\begin{aligned} \frac{dX}{dt} &= a(X) + B(X, X), \\ \frac{dx}{dt} &= a(x) + b(x, X). \end{aligned} \tag{5}$$

The introduction of X has substantially simplified the situation.

Note two important points. First, the equation for X is independent of x_i. Second, the components x_i depend only on X rather than interacting with each other, as occurred in the original system

$$\begin{aligned} \frac{dX}{dt} &= a(X) + B(X, X), \\ \frac{dx}{dt} &= a(x) + b(x, X). \end{aligned} \tag{6}$$

The dimension of this system does not depend on N. This passage is based on the important fact that the

equations for x_i in system (5) are identical. Therefore, the motion of the phase point in the high-dimensional space of $(x_1, x_2, \ldots, x_n)$ is equivalent to the motion of the set of points $(x_1, x_2, \ldots, x_n)$ in the space of only one component. Thus, the limit passage as $N \to \infty$ admits a remarkable interpretation: this is the transition to the phase portrait of system (6) in the space (x, X). This task (the construction of a phase portrait in the low-dimensional space of x) can be successfully handled in modern applied mathematics, which combines sophisticated computational algorithms with analytical methods of the qualitative theory of ordinary differential equations.[1]

MACRODYNAMICS

The first equation

$$\frac{dX}{dt} = a(X) + b(X, X) \tag{7}$$

in the reduced system can be naturally called a macrodynamic equation, because it describes the behavior of the system as a whole. The word macrodynamics was intentionally chosen to be similar to the word thermodynamics. However, we should simultaneously indicate the differences between them. First, thermodynamics is usually associated with asymptotics of a special class of Hamiltonian systems and, in this sense, it is much more delicate and much more sophisticated. Second, despite its name, thermodynamics deals with steady states of systems rather than with their dynamics. Third, the usual thermodynamic approach is axiomatic, and few theorems are available on the properties of passage to the limit as $N \to \infty$. Nevertheless, the main similarity between thermodynamics and macrodynamics is that they both analyze the asymptotic behavior of high-dimensional systems. From this point of view, bilinear systems are fairly useful, because they can be used to test various asymptotic assumptions.

A much wider class of systems exist for which passage to the limit as $N \to \infty$ is also reduced to analyzing phase portraits, but their construction and analysis lie outside the scope of this paper.

Returning to the macrodynamic equation (7), we note that it includes, as special cases, various systems investigated earlier independently, for example, the Verhulst–Pearl equation

$$\frac{dX}{dt} = aX + bX^2; \tag{8}$$

the Volterra system

$$\begin{aligned} \frac{dX}{dt} &= a_1X + b_1XY, \\ \frac{dY}{dt} &= a_2Y + b_2XY; \end{aligned} \tag{9}$$

the Lorenz system

$$\begin{aligned} \frac{dX}{dt} &= -\sigma X + \sigma Y, \\ \frac{dY}{dt} &= \rho X - Y - XZ, \\ \frac{dZ}{dt} &= -\theta Z + XY; \end{aligned} \tag{10}$$

the Lotka system

$$\begin{aligned} \frac{dX}{dt} &= X(\lambda + a_1X + a_2Y + a_3Z), \\ \frac{dY}{dt} &= Y(\mu + b_1X + b_2Y + b_3Z), \\ \frac{dZ}{dt} &= Z(\nu + c_1X + c_2Y + c_3Z), \end{aligned} \tag{11}$$

and other, lesser-known systems, among which we especially note fluid dynamic systems [1, 2] and the Rikitake dynamo [3].

Some remarks should be made about these equations. Verhulst's equation was derived in the middle of the 19th century. It represents a refined Malthus approach to the population dynamics problem. The Volterra system has a similar history. It was also derived in the population dynamics problem (predator–victim) and then found an exact (rather than approximate, as in the original problem) interpretation in chemical kinetics (cold flames). The Lorenz system arose from the hydrodynamic problem of turbulence.

The Lotka system (11) has a curious history. It was derived in an analysis of the kinetics of a simple chemical system and consisted of two equations. Later, it was found that analogous systems of order three arise in population dynamics problems involving two predators and a victim or two victims and a predator. However, the study of this system was long retarded by the lack of computers and the impossibility of an analytical approach. Then systems of this kind were found to arise in the stability analysis of motion (three pairs of purely imaginary roots) even in purely mechanical systems. The well-developed apparatus of stability theory was used to uncover some general properties of these systems. Later, such systems again appeared in connection with Eigen's ideas concerning the chemical evolution (hypercycles) [4], and they were analyzed independently of previous studies. Finally, the complex behavior of such a system (namely, a chain of limit-cycle doublings resulting in the onset of mixing in the system) has recently been studied numerically from an

[1] Some work is for man and the other work is for computer. A mathematician passes to the limit, and then a computer processes data. This is the appropriate division of labor between science and technology.

ecological point of view (human interference in a predator–victim system) by Yu.M. Aponin with coauthors at the Institute of Mathematical Problems of Biology, Russian Academy of Sciences. Steady-state regimes originating in the process of mixing are now known as strange attractors.

The main result of this work is that any of the systems above is interpreted as a reference equation for a multicomponent system. Clearly, this approach considerably increases the range of applicability of all these systems. Another important circumstance is that, in this approach, the systems listed are no longer a random set of historical curious things but become special cases of a uniform general approach; the are reference macrodynamic equations of multicomponent bilinear systems.

The one-dimensional reference equation

$$\frac{dX}{dt} = aX + bX^2 \tag{12}$$

is always the Verhulst–Pearl equation. By making the substitutions

$$X = -\frac{a}{b}\xi, \quad t = \frac{1}{a}\tau \tag{13}$$

it is reduced to the standard form

$$\frac{d\xi}{d\tau} = \xi - \xi^2, \tag{14}$$

which has three fixed points: a point at infinity, $\xi = 0$, and $\xi = 1$. This equation is usually interpreted as a transition from the neighborhood of an unstable fixed point to the neighborhood of the stable fixed point $\xi = 1$.

An escape process $\xi \to -\infty$ is also possible, but this is interpreted as a blowup and system destruction. That is why one-dimensional systems support the idea of a unique macrodynamic equilibrium, which resembles the idea of a thermodynamic equilibrium.

However, the situation differs substantially in the two-dimensional case, where the reference control involves ten parameters:

$$\begin{aligned}\frac{dX_1}{dt} &= a_{11}X_1 + a_{12}X_2 + b_{111}X_1^2 + b_{112}X_1X_2 + b_{122}X_2^2.\\ \frac{dX_2}{dt} &= a_{21}X_1 + a_{22}X_2 + b_{211}X_1^2 + b_{212}X_1X_2 + b_{222}X_2^2.\end{aligned} \tag{15}$$

The variety of types of kinetics arising in this system is much wider but can still be effectively enumerated. A detailed analysis is beyond the scope of this paper. The presence of the exact reference equation describing the behavior of the complete system makes it possible to conjecture that there exists a relationship between phase changes and bifurcations in the macrodynamic equation.

The usual view of phase transitions is associated with changes in steady states (thermodynamic equilibria) depending on the varying external parameters of the system (more frequently, depending on the volume).

However, an exact description of the overall system behavior (by using a macrodynamic equation) makes it possible to extend the concept of a phase change to transitions from one steady-state regime (e.g., a stable equilibrium) to another (e.g., a limit cycle). Moreover, the example of the Lorenz system (or the above-mentioned stochastic regime in the Lotka system) shows that the set of steady-state regimes must include quasi-stochastic regimes (probably, they should better be referred to as mixing modes). Accordingly, the concept of a phase change is extended as well. Clearly, all that was said above holds true for the properties of bilinear systems, and the transfer of this reasoning to Hamiltonian systems requires additional effort.

ASYMPTOTICS OF A REFERENCE EQUATION (IN THE NEIGHBORHOOD OF INFINITY)

A reference equation arises for the variable X, which is the sum of all components and (generally speaking) increases together with N:

$$X = \sum_{k=1}^{N} x_k = N_y. \tag{16}$$

Therefore, the substitution of this relation into (7) and time rescaling

$$t = \frac{1}{N}\tau \tag{17}$$

give the equation

$$\frac{dy}{d\tau} = \frac{1}{N}a(y) + b(y, y), \tag{18}$$

which becomes homogeneous as $N \to \infty$:

$$\frac{dy}{d\tau} = b(y, y). \tag{19}$$

In the special case of Eq. (16), we obtain the two-dimensional system

$$\begin{aligned}\frac{dy_1}{d\tau} &= b_{111}y_1^2 + b_{112}y_1y_2 + b_{122}y_2^2,\\ \frac{dy_2}{d\tau} &= b_{211}y_1^2 + b_{212}y_1y_2 + b_{222}y_2^2.\end{aligned} \tag{20}$$

Dividing one equation by the other gives a homogeneous equation, which, in more usual notation, has the form

$$\frac{dy}{dx} = \frac{ax^2 + bxy + cy^2}{ax^2 + \beta xy + \gamma y^2}. \tag{21}$$

This equation can be integrated by different methods (for example, in polar coordinates).

The qualitative pattern is determined by the number and the character of the invariant rays

$$y = kx, \tag{22}$$

which are found from the following cubic equation implied by (21):

$$k = \frac{a + bk + ck^2}{\alpha + \beta k + \gamma k^2}. \tag{23}$$

The case of complex roots is obtained by merging two rays, which leaves two out of the four possibilities. A more detailed analysis should take into account the linear terms. These terms vanish only in the limit as $X \to \infty$. This passage to the limit is a general case. However, we have to consider another passage to the limit:

$$X \to X_0 \quad \text{as} \quad N \to \infty. \tag{24}$$

This corresponds to the case where the number of x_i increases indefinitely, while their sum remains bounded. Roughly speaking, this is a condition on a larger number of components. The variety of types of kinetics arising under this assumption is much wider that discussed above, but a complete analysis (impossible with no computer) lies outside the scope of this paper.

Finally, we can make one more assumption:

$$X \to 0 \quad \text{as} \quad N \to \infty. \tag{25}$$

In this case, only linear terms remain in the equation. Therefore, the phase portrait of the system reduces to three well-known possibilities: a node, a focus, or a saddle.

CONCLUSIONS

The system involves initial data of three scales. One corresponds to a small neighborhood of a fixed point, and the behavior of the system is described by the linear terms of the reference equation

$$\frac{dX}{dt} = a(X). \tag{26}$$

The second scale corresponds to "finite" values of X. The evolution of such states is described by the full autonomous equation

$$\frac{dX}{dt} = a(X) + b(X, X). \tag{27}$$

In the absence of stable steady-state regimes, the system automatically goes to the following scale (large X). The evolution is described by the highest-order terms of the reference equation

$$\frac{dX}{dt} = b(X, X). \tag{28}$$

All that was said above applies to the case of fixed parameters in the evolution equation. An analogue of the theory of phase changes related to the analysis of phase portrait changes depending on variations in the parameters of the system lies outside the scope of this paper.

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